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Direct Computation of Stochastic Flow in Reservoirs with Uncertain Parameters[1]

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A direct method is presented for determining the uncertainty in reservoir pressure, flow, and net present value (NPV) using the time-dependent, one phase, two- or three-dimensional equations of flow through a porous medium. The uncertainty in the solution is modelled as a probability distribution function and is computed from given statistical data for input parameters such as permeability. The method generates an expansion for the mean of the pressure about a deterministic solution to the system equations using a perturbation to the mean of the input parameters. Hierarchical equations that define approximations to the mean solution at each point and to the field covariance of the pressure are developed and solved numerically. The procedure is then used to find the statistics of the flow and the risked value of the field, defined by the NPV, for a given development scenario. This method involves only one (albeit complicated) solution of the equations and contrasts with the more usual Monte-Carlo approach where many such solutions are required. The procedure is applied easily to other physical systems modelled by linear or nonlinear partial differential equations with uncertain data. © 1997 Academic Press

1. INTRODUCTION

Difficulty in the mathematical and numerical modelling of flow through porous media in underground reservoirs often arises because precise knowledge of the data is not available. Specifically, reservoir data may only be known within certain limits of accuracy, or it may only be possible to specify certain statistical properties of the data. This may be due to inaccuracy in measuring equipment or to inaccessibility and a high level of heterogeneity in the reservoir materials.

The usual approach to problems of this kind is to use Monte-Carlo methods. However, in some cases the number of realisations that need to be generated may be prohibitively large, and for this reason we have aimed to develop a more direct method for assessing the uncertainty in the solution. The procedure described here uses an approach similar to that of [10, 2] and is an extension of techniques that we have previously developed for the stochastic steady-state reservoir flow problem and for a transient mass-balance model with uncertain parameters [3–5]. Preliminary results obtained by this procedure have been published in [6, 7].

The procedure generates an expansion of the mean solution about a deterministic solution to the system equations using a perturbation to the mean of the input parameters. A set of hierarchical equations is obtained for the terms in the expansion of the mean at each point and for the field covariance of the solution. Apart from the deterministic equation, the hierarchical equations are all linear and can be solved sequentially (or, to a large extent, in parallel). This allows a simple, efficient, numerically stable technique to be developed for computing approximations to the mean to any order required. The method can be applied easily to other physical systems governed by linear or nonlinear partial differential equations with stochastic data.

The technique developed here is closely related to that of [9, 13, 14, 11]. In these papers the solution is expanded about its mean, rather than about a deterministic solution, and a set of coupled nonlinear equations for second-order approximations to the mean and covariances of the solutions is derived. These equations are not as easily solved as the hierarchical equations that are established here, and they are not as easily extended to obtain higher order approximations.

The aim of our study is to treat a fairly straightforward two-dimensional model equation for flow through a heterogeneous porous medium (with the implicit assumption that the results obtained may be generalised to the three-dimensional case). The model is derived by combining Darcy's law for flow in a porous medium [8] with the equation for single-phase flow in a fluid with a constant compressibility to give

$$\gamma \frac{\partial p}{\partial t} - \nabla(k \nabla p) = f(\mathbf{r}, t), \quad \mathbf{r} \in D, \tag{1}$$

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and

$$b(p) = 0, \quad \mathbf{r} \in \partial D, \tag{2}$$

where γ is the compressibility, p is the pressure, k is the permeability, $f(\mathbf{r}, t)$ is some forcing function, and $b(p)$ is some general linear differential operator representing either Dirichlet, Neumann, or mixed type boundary conditions. The value of the field is assessed using the net present value (NPV), defined by

$$NPV = \int_0^\infty \|\mathbf{Q}(t)\| e^{-\delta t}\, dt, \tag{3}$$

where $\mathbf{Q}(t)$ is the flow at the relevant production well and δ is some discounting factor.

In the mathematical modelling of the field for a deterministic case, the flow term $\mathbf{Q}(t)$ can easily be obtained if values for the pressure are known or the field flow equations have been solved at each time-step. For the simple model used here, the flow is obtained directly from the formula

$$\mathbf{Q}(t) = -k\nabla(p), \tag{4}$$

where k is the permeability and p is the pressure.

We make the assumption that the statistical behaviour of the permeability field can be characterised by its mean value, $\langle k \rangle$, and the permeability autocorrelation function (PAF), written as a function of two spatial positions, $\mathbf{r}'$ and $\mathbf{r}$. The PAF is defined explicitly as

$$\rho(\mathbf{r}', \mathbf{r}) = \frac{\langle (k(\mathbf{r}') - k_0(\mathbf{r}'))(k(\mathbf{r}) - k_0(\mathbf{r})) \rangle}{\sigma_k(\mathbf{r}')\sigma_k(\mathbf{r})} \tag{5}$$

and can be thought of as a measure of how strongly the statistical properties at points $\mathbf{r}'$ and $\mathbf{r}$ are related. For practical applications, the distribution is assumed to be of a lognormal form. This is a common assumption in groundwater modelling, supported by experimental studies [2].

In the first part of this paper, we deal with cases where uncertainties in the permeabilities cause corresponding uncertainties in the solutions for the pressure. We begin by deriving the hierarchical equations for a general nonlinear problem, using the same notation as in [13, 14]. The technique is then applied to the reservoir flow equations and discrete approximations to the corresponding hierarchical equations are established. Results for a test problem are presented. In the second part of the paper we investigate how the uncertainties in the data propagate into uncertainties in the flow and, more importantly, into the risked value of the field, as measured by the NPV.

2. HIERARCHICAL EQUATIONS FOR GENERAL OPERATOR

We begin by developing a set of hierarchical equations for a general admissible realisation satisfying a general nonlinear evolutionary equation. By developing this system of equations as far as possible and then taking mean values on either side, we obtain equations that allow us to determine the statistical properties of the solution. In the next subsection we apply this technique to derive hierarchical equations for the linear model (1)–(2) for pressure in a reservoir.

We use the notation of MacLaughlin and Wood [13] throughout. We consider an equation of the general form

$$\gamma \frac{\partial p}{\partial t} - \mathscr{F}(p, \theta) = 0, \quad \mathbf{r} \in D, \tag{6}$$

with boundary conditions

$$b(p, \theta) = 0, \quad \mathbf{r} \in \partial D, \tag{7}$$

where

$$p = p(\mathbf{r}, t), \quad \theta = \theta(\mathbf{r}),$$

and the scalar operators $\mathscr{F}$ and b are constructed from spatial derivatives of the dependent variable p. (The operator b is assumed to be linear in p and θ.)

We describe a realisation θ of the data as a perturbation about $\theta_0 = \langle \theta \rangle$ and develop an expansion of the corresponding solution p about p_0, the solution to a deterministic equation which is to be specified. This expansion differs from that of [13], where a perturbation expansion about the mean solution $\langle p \rangle$ is obtained. We let $\theta = \theta_0 + \theta_1$, where $\langle \theta_1 \rangle = 0$, and let $p = \sum_{m=0}^{N} p_m + S_{N+1}$, where the functions $p_m(t)$, $m = 1, 2, \ldots, N$, are to be defined and S_{N+1} is a remainder term due to the truncation of the series. Substituting the entire series into the model equation (6) gives

$$\sum_{m=0}^{N} \gamma \frac{\partial p_m}{\partial t} + \gamma \frac{\partial S_{N+1}}{\partial t} - \mathscr{F}\left(\sum_{m=0}^{N} p_m + S_{N+1}, \theta_0 + \theta_1 \right) = 0. \tag{8}$$

Using the same operator notation as in [13] we perform a Taylor's expansion of $\mathscr{F}$ about (p_0, θ_0), taking the series up to second order ($N = 2$). We find that

$$\sum_{m=0}^{2} \gamma \frac{\partial p_m}{\partial t} + \gamma \frac{\partial S_3}{\partial t} - \mathcal{F}(p_0, \theta_0) - \mathcal{F}_\theta(p_0, \theta_0)\theta_1 - \mathcal{F}_p(p_0, \theta_0)\left(\sum_{m=1}^{2} p_m + S_3\right) - \mathcal{F}_{\theta\theta}(p_0, \theta_0)\frac{\theta_1^2}{2} - \mathcal{F}_{p\theta}(p_0, \theta_0)\theta_1\left(\sum_{m=1}^{2} p_m + S_3\right) - \frac{1}{2}\mathcal{F}_{pp}(p_0, \theta_0)\left(\sum_{m=1}^{2} p_m + S_3\right)\left(\sum_{m=1}^{2} p_m + S_3\right) + R_3 = 0, \quad \mathbf{r} \in D. \tag{9}$$

Collecting all third-order terms together in T_3, we obtain

$$\sum_{m=0}^{2} \gamma \frac{\partial p_m}{\partial t} - \mathcal{F}(p_0, \theta_0) - \mathcal{F}_\theta(p_0, \theta_0)\theta_1 - \mathcal{F}_p(p_0, \theta_0)(p_1 + p_2) - \mathcal{F}_{\theta\theta}(p_0, \theta_0)\frac{\theta_1^2}{2} - \mathcal{F}_{p\theta}(p_0, \theta_0)\theta_1 p_1 - \frac{1}{2}\mathcal{F}_{pp}(p_0, \theta_0)p_1 p_1 + T_3 = 0, \quad \mathbf{r} \in D, \tag{10}$$

with boundary conditions

$$b(p_0, \theta_0) + b_\theta(p_0, \theta_0)\theta_1 + b_p(p_0, \theta_0)(p_1 + p_2 + S_3) = 0, \quad \mathbf{r} \in \partial D. \tag{11}$$

We note that we can also derive an equation for p_m for a general value of N. We find that

$$\sum_{m=0}^{N} \gamma \frac{\partial p_m}{\partial t} - \mathcal{F}(p_0, \theta_0) - \sum_{n=1}^{N}\left(\sum_{i=0}^{n} \frac{\partial^n \mathcal{F}}{\partial p^i\, \partial \theta^{n-i}} \frac{(\sum_{j=1}^{N} p_j)^i \theta_1^{n-i}}{(n-i)!i!}\right) + T_{N+1} = 0, \quad \mathbf{r} \in D, \tag{12}$$

with boundary conditions

$$b(p_0, \theta_0) + b_\theta(p_0, \theta_0)\theta_1 + b_p(p_0, \theta_0)\left(\sum_{m=1}^{N} p_m + S_{N+1}\right) = 0, \quad \mathbf{r} \in \partial D. \tag{13}$$

We now define p_0 to be the deterministic solution satisfying

$$\gamma \frac{\partial p_0}{\partial t} - \mathcal{F}(p_0, \theta_0) = 0, \quad \mathbf{r} \in D, \tag{14}$$

$$b(p_0, \theta_0) = 0, \quad \mathbf{r} \in \partial D, \tag{15}$$

and let p_1 and p_2 be solutions to

$$\gamma \frac{\partial p_1}{\partial t} - \mathcal{F}_\theta(p_0, \theta_0)\theta_1 - \mathcal{F}_p(p_0, \theta_0)p_1 = 0, \quad \mathbf{r} \in D \tag{16}$$

$$b_\theta(p_0, \theta_0)\theta_1 + b_p(p_0, \theta_0)p_1 = 0, \quad \mathbf{r} \in \partial D, \tag{17}$$

and

$$\gamma \frac{\partial p_2}{\partial t} - \mathcal{F}_p(p_0, \theta_0)p_2 - \mathcal{F}_{\theta\theta}(p_0, \theta_0)\frac{\theta_1^2}{2} - \mathcal{F}_{p\theta}(p_0, \theta_0)\theta_1 p_1 - \frac{1}{2}\mathcal{F}_{pp}(p_0, \theta_0)p_1 p_1 = 0, \quad \mathbf{r} \in D, \tag{18}$$

$$p_2 = 0, \quad \mathbf{r} \in \partial D. \tag{19}$$

We can also write down a general equation for p_m, where $m \leq N$, in the form

$$\gamma \frac{\partial p_m}{\partial t} - \sum_{n=1}^{N}\left(\sum_{i=1}^{n} \frac{\partial^n \mathcal{F}}{\partial p^i\, \partial \theta^{n-i}} \frac{n}{(n-1)!i!} \sum_{\Sigma l_k = m-n+i} \Pi_{k=1}^{i} p_{l_k} \theta_1^{n-i}\right) = 0, \tag{20}$$

where the third summation term in Eq. (20) is performed over all possible values of the indices such that the sum of l_k over k equals $m - n + i$. This very complicated term can be simplified in the case where the operator $\mathcal{F}$ is linear in p, due to the fact that all pressure derivatives of $\mathcal{F}$ above the first are then equal to zero. In this case, the general equation for p_m can be written

$$\gamma \frac{\partial p_m}{\partial t} - \sum_{n=1}^{N} \frac{1}{(n-1)!} \frac{\partial^n \mathcal{F}}{\partial p\, \partial \theta^{n-1}} p_{m-n+1} \theta_1^{n-1} = 0. \tag{21}$$

In order to obtain expressions for the mean of p over all possible realisations of the data, we multiply equation (16) by $\theta_1(\mathbf{r}')$ and $p_1(\mathbf{r}', t)$, the values of the first-order perturbations at the point $\mathbf{r}'$, and use the equality

$$\frac{\partial}{\partial t}(p_1(\mathbf{r}', t)p_1(\mathbf{r}, t)) = p_1(\mathbf{r}', t)\frac{\partial p_1(\mathbf{r}, t)}{\partial t} + p_1(\mathbf{r}, t)\frac{\partial p_1(\mathbf{r}', t)}{\partial t}. \tag{22}$$

Then, taking the mean values of the resulting equations and of Eq. (18), we obtain the system

$$\gamma \frac{\partial p_0}{\partial t} - \mathscr{F}(p_0, \theta_0) = 0, \quad \mathbf{r} \in D, \tag{23}$$

$$\gamma \frac{\partial \langle \theta_1(\mathbf{r}') p_1(\mathbf{r}, t)\rangle}{\partial t} - \langle \theta_1(\mathbf{r}') \mathscr{F}_\theta(p_0, \theta_0) \theta_1(\mathbf{r})\rangle - \langle \theta_1(\mathbf{r}') \mathscr{F}_p(p_0, \theta_0) p_1(\mathbf{r}, t)\rangle = 0, \tag{24}$$

$$\gamma \frac{\partial \langle p_1(\mathbf{r}', t) p_1(\mathbf{r}, t)\rangle}{\partial t} - 2\langle p_1(\mathbf{r}', t) \mathscr{F}_\theta(p_0, \theta_0) \theta_1(\mathbf{r})\rangle - 2\langle p_1(\mathbf{r}', t) \mathscr{F}_p(p_0, \theta_0) p_1(\mathbf{r}, t)\rangle = 0, \quad \mathbf{r}', \mathbf{r} \in D, \tag{25}$$

and

$$\gamma \frac{\partial \langle p_2\rangle}{\partial t} - \mathscr{F}_p(p_0, \theta_0)\langle p_2\rangle - \mathscr{F}_{\theta\theta}(p_0, \theta_0) \frac{\langle \theta_1^2\rangle}{2} - \mathscr{F}_{p\theta}(p_0, \theta_0)\langle \theta_1 p_1\rangle - \frac{1}{2}\mathscr{F}_{pp}(p_0, \theta_0)\langle p_1 p_1\rangle = 0, \quad \mathbf{r} \in D, \tag{26}$$

with boundary conditions,

$$b(p_0, \theta_0) = 0, \quad \mathbf{r} \in \partial D, \tag{27}$$

$$\langle \theta_1(\mathbf{r}') b_\theta(p_0, \theta_0) \theta_1(\mathbf{r})\rangle + \langle \theta_1(\mathbf{r}') b_p(p_0, \theta_0) p_1(\mathbf{r}, t)\rangle = 0, \quad \mathbf{r}' \in D, \mathbf{r} \in \partial D, \tag{28}$$

$$\langle p_1(\mathbf{r}', t) b_\theta(p_0, \theta_0) \theta_1(\mathbf{r})\rangle + \langle p_1(\mathbf{r}', t) b_p(p_0, \theta_0) p_1(\mathbf{r}, t)\rangle = 0, \quad \mathbf{r}' \in D, \mathbf{r} \in \partial D, \tag{29}$$

and

$$b_p(p_0, \theta_0)\langle p_2\rangle = 0, \quad \mathbf{r} \in \partial D. \tag{30}$$

Solving Eqs. (23)–(30) then gives a second-order approximation to $\langle p\rangle = p_0 + \langle p_2\rangle + \langle S_3\rangle$. The system equations are triangular and can be solved sequentially. We note that Eqs. (24)–(25) with boundary conditions (28)–(29) are linear and can, in fact, be solved in parallel for different values of $\mathbf{r}'$.

In order to find the equations up to a general mth-order term, we take mean values on either side of Eq. (20). For the case where the operator is linear in p, we obtain

$$\gamma \frac{\partial \langle p_m\rangle}{\partial t} - \sum_{n=1}^{N} \frac{1}{(n-1)!} \frac{\partial^n \mathscr{F}}{\partial p\, \partial \theta^{n-1}} \langle p_{m-n+1} \theta_1^{n-1}\rangle = 0, \quad \mathbf{r} \in D, \tag{31}$$

and

$$b_p(p_0, \theta_0)\langle p_m\rangle = 0, \quad \mathbf{r} \in \partial D, \tag{32}$$

for $m \geq 2$. Evolutionary equations for the cross-correlation terms $\langle p_{m-n+1} \theta_1^{n-1}\rangle$ are obtained by the same process as in the case $N = 2$.

It is important to note that the full hierarchical system of equations for the terms up to mth order in the expansion of $\langle p\rangle$ obtained by this process also has a triangular structure. This is vital in the development of the numerical procedure for computing the mean and covariances of p.

It is also important to note here that, apart from the deterministic equation for p_0, the hierarchical equations are all linear differential equations. The basic triangular structure of the system therefore allows all the unknown terms to be computed, in principle, by the application of a nonlinear solver to the deterministic equation, followed by the application of a basic linear solver to each of the remaining hierarchical equations, up to the approximation order required. It is assumed that a method for solving the deterministic problem is already available, and thus, the only extra work involved in determining the stochastic mean and covariances arises in the solution of the higher-order linear equations.

We next consider the application of this procedure to the model problem (1)–(2) for flow in a porous medium, where the differential operator is linear in p.

3. HIERARCHICAL EQUATIONS FOR POROUS FLOW PROBLEM

We now apply the theory developed in the previous section to the model problem (1)–(2). If the permeability is assumed to have a symmetric probability distribution, then we take $\theta \equiv k$ and let the operator have the form

$$\mathscr{F}(p, k) = \nabla k\, \nabla p. \tag{33}$$

In practice we assume that the permeability has a lognormal distribution, which is the common form for a porous medium [2, 12]. We take $\theta \equiv z$, where $\ln(k) = z = z_0 + z_1$ and let the operator be given by

$$\mathscr{F}(p, z) = \nabla e^z\, \nabla p. \tag{34}$$

In this case we expand the permeability about the geometric mean. Then $\ln(k) = z_0 + z_1$, where $z_0 = \langle z\rangle$, implies

$$\begin{aligned} k &= e^{z_0} + z_1 e^{z_0} + \frac{z_1^2}{2} e^{z_0} + \cdots \\ &= \kappa_g + \kappa_1 + \kappa_2 + \cdots = \kappa_g + \sum_{j=1}^{\infty} \kappa_j, \end{aligned} \tag{35}$$

where κ_g is the geometric mean.

If we perform the same procedure as in Section 2 and assume that the pressure has the form

$$p = \sum_{m=0}^{N} p_m + S_{N+1}, \tag{36}$$

then substituting for pressure and permeability into Eq. (1) gives

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}\left(\sum_{m=0}^{N} p_m + S_{N+1}\right) \\ &\quad - \nabla\left(\kappa_g + \sum_{j=1}^{\infty} \kappa_j\right) \nabla\left(\sum_{m=0}^{N} p_m + S_{N+1}\right) = f(\mathbf{r}, t). \end{aligned} \tag{37}$$

Writing

$$f(\mathbf{r}, t) = f_0(\mathbf{r}, t) + f_1(\mathbf{r}, t),$$

where $f_0 = \langle f(\mathbf{r}, t)\rangle$ and f_1 is a perturbation with mean value equal to zero, we obtain the system of hierarchical equations

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \kappa_g \nabla p_0 = f_0, \tag{38}$$

$$\gamma \frac{\partial p_1}{\partial t} - \nabla \kappa_g \nabla p_1 - \nabla \kappa_1 \nabla p_0 = f_1, \tag{39}$$

$$\gamma \frac{\partial p_2}{\partial t} - \nabla \kappa_g \nabla p_2 - \nabla \kappa_1 \nabla p_1 - \nabla \kappa_2 \nabla p_0 = 0, \tag{40}$$

$$\vdots$$

$$\gamma \frac{\partial p_j}{\partial t} - \nabla \kappa_g \nabla p_j - \sum_{m=0}^{j-1} \nabla \kappa_{j-m} \nabla p_m = 0, \tag{41}$$

$$\vdots$$

$$\gamma \frac{\partial p_N}{\partial t} - \nabla \kappa_g \nabla p_N - \sum_{m=0}^{N-1} \nabla \kappa_{N-m} \nabla p_m = 0, \tag{42}$$

$$\begin{aligned} &\gamma \frac{\partial S_{N+1}}{\partial t} - \nabla \kappa_g \nabla S_{N+1} - \nabla\left(\sum_{j=1}^{\infty} \kappa_j\right) \nabla S_{N+1} \\ &\quad - \sum_{j=N+1}^{\infty} \sum_{m=0}^{N} \nabla \kappa_{j-m} \nabla p_m = 0. \end{aligned} \tag{43}$$

Taking mean values on either side of these equations and performing the same procedure on Eq. (39) as in Section 2 (with $N = 2$), we obtain, up to second order, the hierarchical system

$$\gamma \frac{\partial p_0}{\partial t} - \nabla \kappa_g \nabla p_0 = f_0 \tag{44}$$

$$\begin{aligned} &\gamma \frac{\partial \langle \kappa_1(\mathbf{r}') p_1(\mathbf{r}, t)\rangle}{\partial t} - \nabla \kappa_g(\mathbf{r}) \nabla \langle \kappa_1(\mathbf{r}') p_1(\mathbf{r}, t)\rangle \\ &\quad - \nabla \langle \kappa_1(\mathbf{r}') \kappa_1(\mathbf{r})\rangle \nabla p_0(\mathbf{r}, t) = \langle \kappa_1(\mathbf{r}') f_1(\mathbf{r})\rangle, \end{aligned} \tag{45}$$

$$\gamma \frac{\partial \langle p_2\rangle}{\partial t} - \nabla \kappa_g \nabla \langle p_2\rangle - \nabla \langle \kappa_1 \nabla p_1\rangle - \nabla \langle \kappa_2\rangle \nabla p_0 = 0, \tag{46}$$

where

$$\mathbf{r}' \in D, \mathbf{r} \in D,$$

with the boundary conditions

$$b(p_0(\mathbf{r}, t)) = 0, \tag{47}$$

$$b_p(p_0)\langle \kappa_1(\mathbf{r}') p_1(\mathbf{r}, t)\rangle = 0, \tag{48}$$

$$b_p(p_0)\langle p_2(\mathbf{r}, t)\rangle = 0, \tag{49}$$

where

$$\mathbf{r}' \in D, \mathbf{r} \in \partial D.$$

We may also obtain a second-order approximation to the covariance as in the previous section by considering

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(p_1(\mathbf{r}', t) p_1(\mathbf{r}, t)) \\ &\quad = p_1(\mathbf{r}', t)\gamma \frac{\partial p_1(\mathbf{r}, t)}{\partial t} + p_1(\mathbf{r}, t)\gamma \frac{\partial p_1(\mathbf{r}', t)}{\partial t} \end{aligned} \tag{50}$$

and substituting for $\gamma(\partial p_1/\partial t)$, etc. from (39) to obtain

$$\begin{aligned} &\gamma \frac{\partial}{\partial t}(p_1(\mathbf{r}', t) p_1(\mathbf{r}, t)) \\ &\quad - \nabla_2 \kappa_g(\mathbf{r}) \nabla_2 p_1(\mathbf{r}', t) p_1(\mathbf{r}, t) \\ &\quad - \nabla_2 \kappa_1(\mathbf{r}) p_1(\mathbf{r}', t) \nabla_2 p_0(\mathbf{r}, t) \\ &\quad - \nabla_1 \kappa_g(\mathbf{r}') \nabla_1 p_1(\mathbf{r}, t) p_1(\mathbf{r}', t) \\ &\quad - \nabla_1 \kappa_1(\mathbf{r}') p_1(\mathbf{r}, t) \nabla_1 p_0(\mathbf{r}', t) = 0, \end{aligned} \tag{51}$$

where ∇_1 and ∇_2 denote the grad with respect to $\mathbf{r}'$ and $\mathbf{r}$, respectively. Taking the mean value on either side of this equation results in an equation for the behaviour of the covariance of the pressure given by

$$\begin{aligned}\gamma \frac{\partial}{\partial t}(\langle p_1(\mathbf{r}', t) p_1(\mathbf{r}, t)\rangle) &\\ - \nabla_2 \kappa_g(\mathbf{r}, t)\, \nabla_2 \langle p_1(\mathbf{r}', t) p_1(\mathbf{r}, t)\rangle &\\ - \nabla_2 \langle \kappa_1(\mathbf{r}) p_1(\mathbf{r}', t)\rangle\, \nabla_2 p_0(\mathbf{r}, t) &\\ - \nabla_1 \kappa_g(\mathbf{r}')\, \nabla_1 \langle p_1(\mathbf{r}, t) p_1(\mathbf{r}', t)\rangle &\\ - \nabla_1 \langle \kappa_1(\mathbf{r}') p_1(\mathbf{r}, t)\rangle\, \nabla_1 p_0(\mathbf{r}', t) = 0. &\end{aligned} \tag{52}$$

If the covariance at time t between pressure values at two points, $\mathbf{r}'$ and $\mathbf{r}$, is denoted by $C(\mathbf{r}', \mathbf{r}, t)$, then these equations are

$$\begin{aligned}\gamma \frac{\partial}{\partial t}(C(\mathbf{r}', \mathbf{r}, t)) &\\ - \nabla_2 \kappa_g(\mathbf{r}, t)\, \nabla_2 C(\mathbf{r}', \mathbf{r}, t) &\\ - \nabla_2 \langle \kappa_1(\mathbf{r}) p_1(\mathbf{r}', t)\rangle\, \nabla_2 p_0(\mathbf{r}, t) &\\ - \nabla_1 \kappa_g(\mathbf{r}')\, \nabla_1 C(\mathbf{r}, \mathbf{r}', t) &\\ - \nabla_1 \langle \kappa_1(\mathbf{r}') p_1(\mathbf{r}, t)\rangle\, \nabla_1 p_0(\mathbf{r}', t) = 0, &\end{aligned} \tag{53}$$

with the boundary conditions,

$$b_p(p_0) C(\mathbf{r}', \mathbf{r}, t) = 0, \quad \mathbf{r}' \in D, \mathbf{r} \in \partial D, \tag{54}$$

In summary, the solutions to Eqs. (44), (45), (46), and (53) with boundary conditions (47)–(49) and (54) define a second-order approximation to the mean and covariance of the pressure p. In order to solve these equations numerically, the cross-correlation term, $\nabla\langle \kappa_1\ \nabla p_1\rangle$, in Eq. (45) must be treated with special care. In the next section we discuss a suitable scheme for discretising the hierarchical equations.

Higher order approximations can be obtained by solving the hierarchical system for higher order terms, $\langle p_m\rangle$, $m = 3, 4, ..., N$, in the expansion of $\langle p\rangle$. The accuracy of the approximation is determined by the truncation error S_{N+1}. It is possible that bounds can be obtained on the size of this remainder term over all admissible realizations. In [4, 5] an analysis of the truncation error is given for the steady-state problem and bounds are derived in terms of bounds on the range of possible values for the permeability. This effectively gives a measure of the accuracy of the hierarchical approximations in the limit as the system tends to steady-state. It is expected that this analysis can be extended to the transient case.

4. DISCRETISATION

We now show that the problem of treating $\langle \kappa_1\ \nabla p_1\rangle$ numerically may be overcome by discretising the hierarchical equations derived in Section 3 in an appropriate way.

4.1. *General Form of the Discrete Equations*

We use a simple explicit time scheme and a general (unspecified) spatial difference scheme of the form

$$\frac{\gamma p_{0\,ij}^{n+1} - \gamma p_{0\,ij}^{n}}{\Delta t} - \nabla_h(\kappa_{ij}^{g}\, \nabla_h p_{0\,ij}^{n}) = \tilde{f}_{0\,ij}^{n},$$

where the term $\tilde{f}_{0\,ij}^{n}$ includes implicitly both the forcing function f from the analytic equations and the associated boundary conditions at points adjacent to or on the boundary.

We discretise Eqs. (39) and (40) correspondingly and then multiply the discretisation of Eq. (39) at the grid point (i, j) by the value of κ_1 at some other grid point (i', j'). Taking the mean value on either side of the resultant equations gives the hierarchical set of discrete equations

$$\frac{\gamma p_{0\,ij}^{n+1} - \gamma p_{0\,ij}^{n}}{\Delta t} - \nabla_h(\kappa_{ij}^{g}\, \nabla_h p_{0\,ij}^{n}) = \tilde{f}_{0\,ij}^{n}, \tag{55}$$

$$\begin{aligned}&\frac{\gamma\langle \kappa_{i'j'}^{1} p_{1\,ij}^{n+1}\rangle - \gamma\langle \kappa_{i'j'}^{1} p_{1\,ij}^{n}\rangle}{\Delta t}\\ &\quad - \nabla_h(\kappa_{ij}^{g}\, \nabla_h \langle \kappa_{i'j'}^{1} p_{1\,ij}^{n}\rangle)\\ &\quad - \nabla_h(\langle \kappa_{i'j'}^{1} \kappa_{ij}^{1}\rangle\, \nabla_h p_{0\,ij}^{n}) = \langle \kappa_{i'j'}^{1} \tilde{f}_{1\,ij}^{n}\rangle,\end{aligned} \tag{56}$$

$$\begin{aligned}&\frac{\gamma\langle p_{2\,ij}^{n+1}\rangle - \gamma\langle p_{2\,ij}^{n}\rangle}{\Delta t}\\ &\quad - \nabla_h(\kappa_{ij}^{g}\, \nabla_h \langle p_{2\,ij}^{n}\rangle) - \nabla_h(\langle \kappa_{ij}^{1}\, \nabla_h p_{1\,ij}^{n}\rangle)\\ &\quad - \nabla_h(\langle \kappa_{ij}^{2}\rangle\, \nabla_h p_{0\,ij}^{n}) = 0,\end{aligned} \tag{57}$$

where the indices (i, j) refer to spatial points $(i\ \Delta x, j\ \Delta y)$ in Cartesian coordinates, and $p_{m\,ij}^{n}$ refers to the numerical solution for $p_m(\mathbf{r}, n\ \Delta t)$, where $\mathbf{r}$ is also in Cartesian coordinates.

The same discretisation performed on the covariance equation (53) results in the equations

$$\begin{aligned}&\frac{\gamma C_{i'j'ij}^{n+1} - \gamma C_{i'j'ij}^{n}}{\Delta t}\\ &\quad - \nabla_h \kappa_{ij}^{g}\, \nabla_h C_{i'j'ij}^{n} - \nabla_h \langle \kappa^1 p_1\rangle_{i'j'ij}^{n}\, \nabla_h p_{0\,ij}^{n}\\ &\quad - \nabla_h \kappa_{i'j'}^{g}\, \nabla_h C_{iji'j'}^{n} - \nabla_h \langle \kappa^1 p_1\rangle_{iji'j'}^{n}\, \nabla_h p_{0\,i'j'}^{n} = 0.\end{aligned} \tag{58}$$

The quantity of particular interest is the variance of the pressure distribution, an important characterisation of the complete distribution function. In discretised form, the variance for time level $n\,\Delta t$, at spatial position $(i\,\Delta x, j\,\Delta y)$ is the value of C^n_{ijij}. Unfortunately, in the process of solving for this value, the correlation values for distinct points, $C^n_{i'j'ij}$ must also be computed and stored at each time level. These values can be considered as a bonus to the required information, having an academic rather than a practical interest. An indication of the correlation length of the solution variable is, however, now directly available through this technique.

We now have a complete set of coupled numerical equations for approximating the first two moments of the probability distribution of the pressure up to second order. The boundary conditions are incorporated into the right-hand side terms of the equations. (Higher order approximations can be obtained by discretising the higher order equations in the full hierarchical system.) When these equations are solved simultaneously, the cross-correlation terms are found from the linear equation (56) and then substituted into Eq. (57). It is important to note that in order to obtain the correct spatial discretisation of (57), it is necessary to discretise the term $\nabla_h(\langle \kappa^1_{ij}\,\nabla_h p^n_{1\,ij}\rangle)$ in Eq. (57) *first* and then to substitute the solutions $\langle \kappa^1_{i'j'} p^n_{1\,ij}\rangle$ to Eqs. (56) into the resulting numerical expressions. In the next subsection we describe a specific spatial discretisation that gives a stable, second-order numerical scheme.

4.2. *Spatial Discretization*

We now describe a specific spatial discretisation of the hierarchical equations.

We consider a simple explicit five-point difference scheme, where the value of the permeability at points halfway between adjacent gridpoints (i, j) and $(i \pm 1, j)$ or $(i, j \pm 1)$ is always approximated by an average of the two values at the grid-points. Equation (55) at an interior grid point in this case becomes

$$\begin{aligned}
&\frac{\gamma p^{n+1}_{0\,ij} - \gamma p^n_{0\,ij}}{\Delta t} + \frac{(\kappa^g_{i+1j} + \kappa^g_{ij})}{2\,\Delta x^2} p^n_{0\,i+1j} + \frac{(\kappa^g_{i-1j} + \kappa^g_{ij})}{2\,\Delta x^2} p^n_{0\,i-1j} \\
&\quad + \frac{(\kappa^g_{ij+1} + \kappa^g_{ij})}{2\,\Delta y^2} p^n_{0\,ij+1} + \frac{\kappa^g_{ij-1} + \kappa^g_{ij})}{2\,\Delta y^2} p^n_{0\,ij-1} \\
&\quad - \left\{\frac{(\kappa^g_{i+1j} + \kappa^g_{i-1j} + 2\kappa^g_{ij})}{2\,\Delta x^2} + \frac{(\kappa^g_{ij+1} + \kappa^g_{ij-1} + 2\kappa^g_{ij})}{2\,\Delta y^2}\right\} p^n_{0\,ij} = f^n_{0\,ij}.
\end{aligned} \tag{59}$$

Equations (56)–(58) are discretised similarly. The term $\nabla_h(\langle \kappa^1_{ij}\,\nabla_h p^n_{1\,ij}\rangle)$ in Eq. (57) is approximated using the computed solutions $\langle \kappa^1_{i'j'} p^n_{1\,ij}\rangle$ to (56) at points (i', j') adjacent to the discretisation point (i, j). The complete discrete form of equation (57) is given by

$$\begin{aligned}
&\frac{\gamma\langle p^{n+1}_{2\,ij}\rangle - \gamma\langle p^n_{2\,ij}\rangle}{\Delta t} + \frac{(\kappa^g_{i+1j} + \kappa^g_{ij})}{2\,\Delta x^2}\langle p^n_{2\,i+1j}\rangle + \frac{(\kappa^g_{i-1j} + \kappa^g_{ij})}{2\,\Delta x^2}\langle p^n_{2\,i-1j}\rangle \\
&\quad + \frac{(\kappa^g_{ij+1} + \kappa^g_{ij})}{2\,\Delta y^2}\langle p^n_{2\,ij+1}\rangle + \frac{(\kappa^g_{ij-1} + \kappa^g_{ij})}{2\,\Delta y^2}\langle p^n_{2\,ij-1}\rangle \\
&\quad - \left\{\frac{(\kappa^g_{i+1j} + \kappa^g_{i-1j} + 2\kappa^g_{ij})}{2\,\Delta x^2} + \frac{(\kappa^g_{ij+1} + \kappa^g_{ij-1} + 2\kappa^g_{ij})}{2\,\Delta y^2}\right\}\langle p^n_{2\,ij}\rangle \\
&\quad + \frac{(\langle\kappa^1 p_1\rangle^n_{i+1ji+1j} + \langle\kappa^1 p_1\rangle^n_{iji+1j})}{2\,\Delta x^2} + \frac{(\langle\kappa^1 p_1\rangle^n_{i-1ji-1j} + \langle\kappa^1 p_1\rangle^n_{iji-1j})}{2\,\Delta x^2} \\
&\quad + \frac{(\langle\kappa^1 p_1\rangle^n_{ij+1ij+1} + \langle\kappa^1 p_1\rangle^n_{ijij+1})}{2\,\Delta y^2} + \frac{(\langle\kappa^1 p_1\rangle^n_{ij-1ij-1} + \langle\kappa^1 p_1\rangle^n_{ijij-1})}{2\,\Delta y^2} \\
&\quad - \frac{(\langle\kappa^1 p_1\rangle^n_{i+1jij} + \langle\kappa^1 p_1\rangle^n_{i-1jij} + 2\langle\kappa^1 p_1\rangle^n_{ijij})}{2\,\Delta x^2} \\
&\quad - \frac{(\langle\kappa^1 p_1\rangle^n_{ij+1ij} + \langle\kappa^1 p_1\rangle^n_{ij-1ij} + 2\langle\kappa^1 p_1\rangle^n_{ijij})}{2\,\Delta y^2} \\
&\quad + \frac{(\kappa^2_{i+1j} + \kappa^2_{ij})}{2\,\Delta x^2} p^n_{0\,i+1j} + \frac{(\kappa^2_{i-1j} + \kappa^2_{ij})}{2\,\Delta x^2} p^n_{0\,i-1j} \\
&\quad + \frac{(\kappa^2_{ij+1} + \kappa^2_{ij})}{2\,\Delta y^2} p^n_{0\,ij+1} + \frac{(\kappa^2_{ij-1} + \kappa^2_{ij})}{2\,\Delta y^2} p^n_{0\,ij-1} \\
&\quad - \left\{\frac{(\kappa^2_{i+1j} + \kappa^2_{i-1j} + 2\kappa^2_{ij})}{2\,\Delta x^2} + \frac{(\kappa^2_{ij+1} + \kappa^2_{ij-1} + 2\kappa^2_{ij})}{2\,\Delta y^2}\right\} p^n_{0\,ij} = 0.
\end{aligned} \tag{60}$$

Provided that the pressure is specified as a deterministic function of time at one point in the region or on its boundary, it can be shown that the complete numerical scheme is *stable* if the condition

$$\frac{4\,\Delta t\,\kappa_g}{\gamma h^2} < 1 \tag{61}$$

holds, where $h = \Delta x = \Delta y$. In practice the pressure at a well site is controlled and, therefore, the assumption that the pressure is specified deterministically at some point is a natural constraint on the system.

If the pressure is not specified as a deterministic function at some point in the region or on its boundary (i.e., purely Neumann boundary conditions are specified), then the approximation to the deterministic equation for p_0 is stable under the condition (61), but the numerical scheme for the complete hierarchical equations is unconditionally *unstable* and errors are expected to propagate with a polynomial growth rate.

5. APPLICATION

In this section we present examples illustrating the results obtained by this method for the full statistical problem.

5.1. *Test Problem*

We consider the system equations (1)–(2) on a square of unit length with centre at (0.5, 0.5). We assume the parameter γ is deterministic with unit value and the forcing function f is deterministic and identically zero; that is, $\gamma = 1$, and $f \equiv 0$. No flow conditions are assumed around the boundary. It follows that

$$b(p) = \partial p/\partial n, \quad b_p(p_0)(\cdot) = \partial(\cdot)/\partial n.$$

A single Fourier mode is taken as the initial condition for the pressure, and the initial values for the mean and variance of the pressure are zero throughout the region (equivalent to a deterministic initial condition). The pressure at the centre of the region is assumed to be deterministic and is held fixed at a value of zero for all time; the higher moments are, thus, also zero at this point for all time. The boundary conditions for each realization are given explicitly by

$$\frac{\partial p_m}{\partial n} = 0, \quad x \in [0,1], \, y = 0, 1$$

$$\frac{\partial p_m}{\partial n} = 0, \quad y \in [0,1], \, x = 0, 1,$$

together with the conditions

$$p_m(x, y, t) = 0, \quad \text{at } x = 0.5, \, y = 0.5,$$

and the initial conditions are given by

$$p_0(x, y, 0) = \cos(\pi x), \quad x, y \in [0, 1],$$
$$p_m(x, y, 0) = 0, \qquad x, y \in [0, 1],$$

for $m = 0, 1, 2, \ldots$.

All lengths and times are normalised. It is assumed here that one unit of length corresponds to one kilometre. If one unit of time is taken to represent 10 years, then one pressure unit corresponds to 450 psi.

Using a single Fourier mode as the initial condition implies that in the case of a homogeneous geometric mean value, κ_g, for the permeability, the solution to the deterministic equation (55) may be expressed as the Fourier mode

$$p_0(x, y, t) = e^{-\pi^2(\kappa_g/\gamma)t} \cos(\pi x) \tag{62}$$

with an exponentially decaying amplitude. It is fairly trivial to show by substitution that (62) is a solution to the model equation satisfying the boundary conditions. We choose this test function as it is a straightforward solution with well-known deterministic behaviour.

TABLE I

Numerical Solution for p_0 with Isotropic Correlation Lengths

N	p_0(**a**)	p_0(**b**)	p_0(**c**)	p_0(**d**)
4	0.16	−0.18	−0.18	0.16
8	0.22	−0.22	−0.22	0.22
12	0.24	−0.25	−0.25	0.24
16	0.26	−0.26	−0.26	0.26
18	0.26	−0.26	−0.26	0.26

Note. $\kappa_g = 0.2$, $\sigma_z = 0.10$, $\Delta t/h^2 = 0.648$, and $h = 1/N$.

In the experiments presented here, the values for the geometric mean of the permeability and for the variance of the log of the permeability, $z \equiv \ln(k)$, are taken to be constants, $\kappa_g \equiv e^{\langle z \rangle} = 0.2$ and $\sigma_z = 0.1$, respectively. The PAF of the log of the permeability is given by

$$\rho(x, y, x', y') = \exp\left(-\left(\frac{\pi}{2}\frac{(x-x')}{\lambda_x}\right)^2\right) \exp\left(-\left(\frac{\pi}{2}\frac{(y-y')}{\lambda_y}\right)^2\right). \tag{63}$$

Both the isotropic case where $\lambda_x = \lambda_y$, and the anisotropic case, $\lambda_x \neq \lambda_y$, are considered.

5.2. *Results*

In Tables I–III we present a brief summary of results demonstrating the convergence of the computational scheme. The numerical solutions of the deterministic pressure, the corrected value to the pressure mean and the covariance are shown at four different points in the reservoir after a time of $t = 1.0$. The points are labelled **a**, **b**, **c**, and **d**, where **a** = (0.25, 0.25), b = (0.75, 0.25), **c** = (0.75, 0.75), and **d** = (0.25, 0.75). In these tests we let $\mu = \Delta t/h^2 = 0.648$ remain constant as $h = 1/N$ converges to 0. The tables indicate that the full numerical hierarchical equations are convergent and that about three figures of accuracy can

TABLE II

Numerical Solution for p_2 with Isotropic Correlation Lengths

N	p_2(**a**)	p_2(**b**)	p_2(**c**)	p_2(**d**)
4	4.05×10^{-2}	4.32×10^{-2}	4.23×10^{-2}	4.20×10^{-2}
8	3.64×10^{-2}	4.09×10^{-2}	4.04×10^{-2}	3.86×10^{-2}
12	3.47×10^{-2}	3.95×10^{-2}	3.99×10^{-2}	3.72×10^{-2}
16	3.42×10^{-2}	3.87×10^{-2}	3.92×10^{-2}	3.66×10^{-2}
18	3.44×10^{-2}	3.90×10^{-2}	3.91×10^{-2}	3.69×10^{-2}

Note. $\kappa_g = 0.2$, $\sigma_z = 0.10$, $\Delta t/h^2 = 0.648$, and $h = 1/N$.

TABLE III

Numerical Solution for Cov_p with Isotropic Correlation Lengths

N	$\mathrm{Cov}_p(\mathbf{a})$	$\mathrm{Cov}_p(\mathbf{b})$	$\mathrm{Cov}_p(\mathbf{a})$	$\mathrm{Cov}_p(\mathbf{d})$
4	2.85×10^{-4}	2.80×10^{-4}	3.43×10^{-4}	3.59×10^{-4}
8	3.02×10^{-4}	3.08×10^{-4}	3.71×10^{-4}	3.82×10^{-4}
12	3.18×10^{-4}	3.21×10^{-4}	3.86×10^{-4}	3.91×10^{-4}
16	3.24×10^{-4}	3.27×10^{-4}	3.91×10^{-4}	4.01×10^{-4}
18	3.25×10^{-4}	3.27×10^{-4}	3.92×10^{-4}	4.03×10^{-4}

Note. $\kappa_g = 0.2$, $\sigma_z = 0.10$, $\Delta t/h^2 = 0.648$, and $h = 1/N$.

be expected with a value of $N = 18$. Additional studies relating to the stability of the scheme show also that with this value of N, the solution is not significantly improved by taking any smaller values of Δt.

In Figs. 1–10 the solutions to the hierarchical equations computed with $h = \frac{1}{18}$ and $\Delta t = \frac{1}{500}$ are shown. The data is defined in Section 5.1. Experiments have also been carried out with different choices for the mean and the variance of the probability distribution of the permeability and for different time intervals.

In Figs. 1 and 2, we show the evolution of the deterministic pressure solution, first at time $t = 0.1$ and then at the final time value $t = 1.0$. Figures 3 and 4 then show the correction for the mean at these two times and Figs. 5 and 6 demonstrate the values of the variance at the same time points.

The next set of four figures shows the case where the correlation lengths are anisotropic. Figures 7 and 8 show plots at the final time, where the correlation length is short in the x-direction, and long in the y-direction, with $\lambda_x = 0.1$ and $\lambda_y = 1.0$. The plots are for the mean

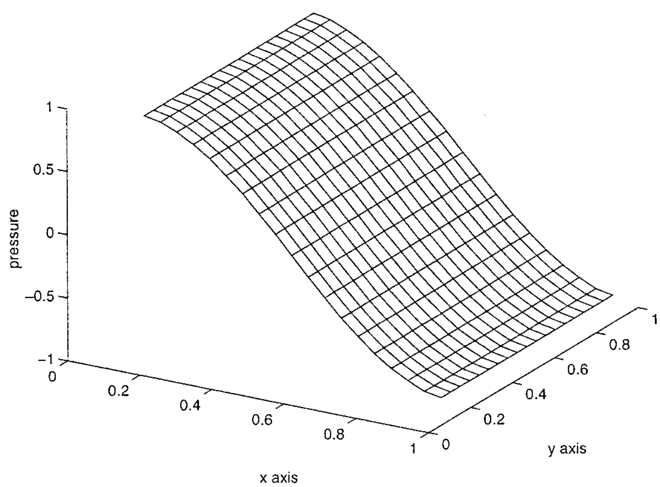

FIG. 1. Deterministic solution for pressure at $t = 0.1$.

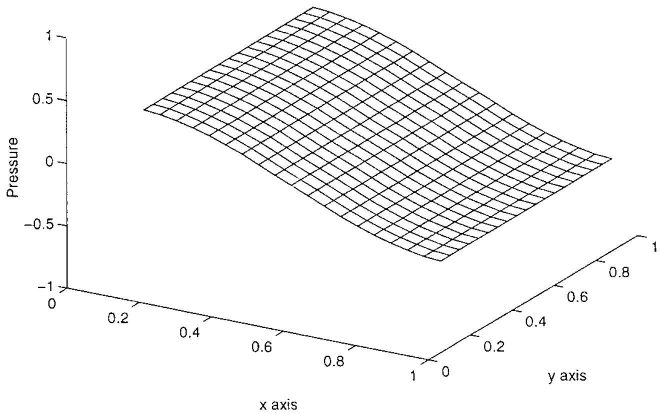

FIG. 2. Deterministic solution for pressure at $t = 1.0$.

correction to the deterministic solution and for the variance, respectively, after time interval $t = 1.0$. Figures 9 and 10 show plots for the same values at $t = 1.0$, but with anisotropic correlation lengths reversed, so that $\lambda_x = 1.0$ and $\lambda_y = 0.1$.

5.3. *Discussion*

The deterministic solution behaves as expected, decaying exponentially whilst retaining the basic shape of the (one-dimensional) mode. The numerical amplitude at time $t = 1.0$ is 0.140, compared to the analytic value of $e^{-\pi^2 \times 0.2} = 0.139$.

We can see in Figs. 3–6 how the statistical moments grow from very low values, close to zero at the initial time to more significant values at the final time. This is to be expected as the initial conditions are assumed to be deterministic and the statistical moments are zero at $t = 0$.

The variance is seen to reach a maximum at around $t = 0.5$, thereafter gradually decreasing, with the maximum concentrating in the corners as it decays.

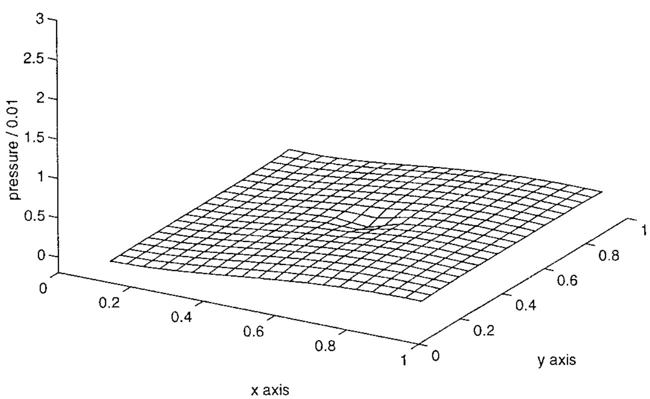

FIG. 3. Mean correction to the deterministic pressure at $t = 0.1$.

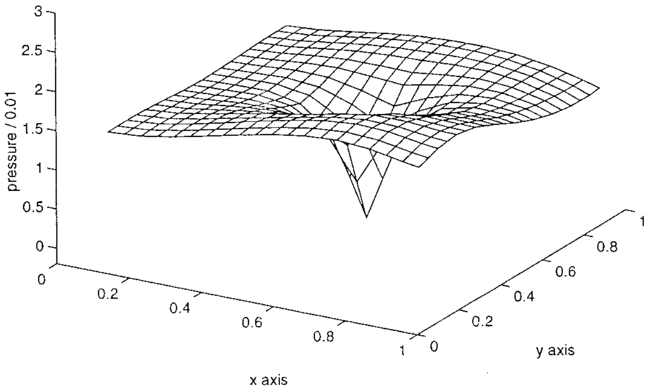

FIG. 4. Mean correction to the deterministic pressure at t = 1.0.

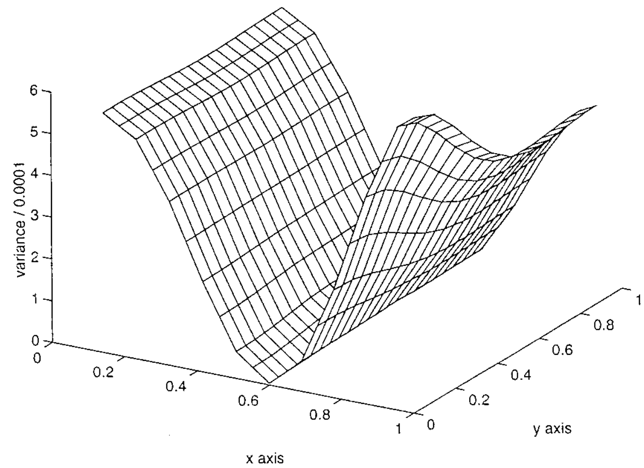

FIG. 6. Pressure variance at t = 1.0.

In comparison with experiments using a higher mean value, we observe a slower decay rate; for example, when $\kappa_g = 0.1$, the numerical decay rate is halved. The general shape assumed by the approximations to the mean and variance of p after one time unit are the same. The numerical value of the variance is, however, higher due to a greater relative spread in admissible realisations.

In the case of strong correlation in the y-direction, and much less correlation in the x-direction, we find that the statistical properties throughout the region are more homogeneous in themselves than in the case where the strong correlation is in the x-direction, and there are much higher variances concentrated in the corners. In the case where we consider small isotropic correlation lengths in both directions we observe a similar concentration of variance in the corners, with numerical values of one order of magnitude lower, which is the type of behaviour we expect if the statistical properties are weakly correlated.

6. TREATMENT OF THE FLUID FLOW AND NPV

6.1. *Fluid Flow*

The equation for flow in a porous medium can be obtained from the pressure in the fluid using Darcy's law, which is given in simplest form by

$$\mathbf{Q} = -k\,\nabla p. \tag{64}$$

In the case of a lognormal probability distribution we may substitute the perturbation expansion (35) for the permeability into Eq. (64). Assuming, as previously, that the pressure may be approximated by a truncated series of form (36), we find

$$\mathbf{Q} \simeq -(\kappa_g + \kappa_1 + \kappa_2)\,\nabla(p_0 + p_1 + p_2), \tag{65}$$

where all terms up to and including second order have been retained.

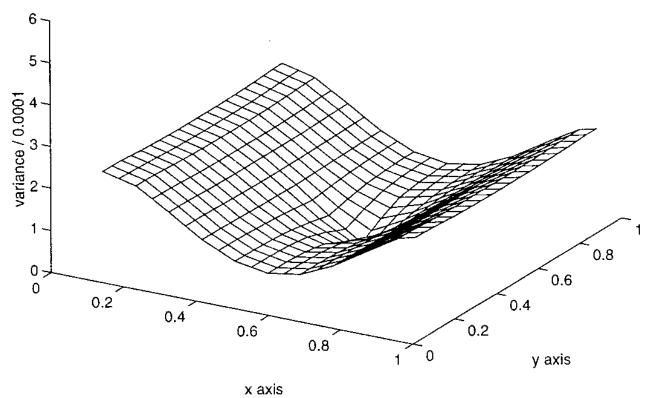

FIG. 5. Pressure variance t = 0.1.

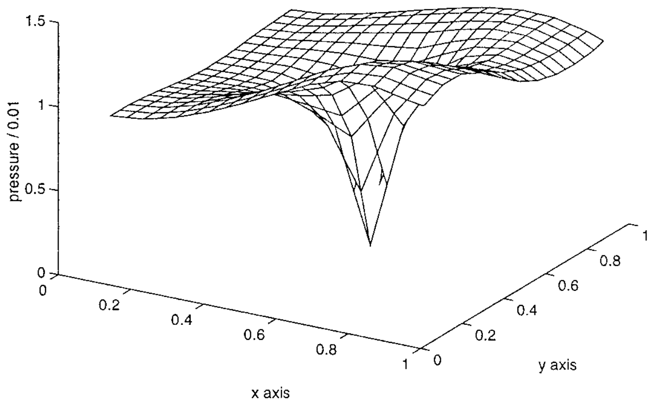

FIG. 7. Mean correction for anisotropic correlation lengths at t = 1.0.

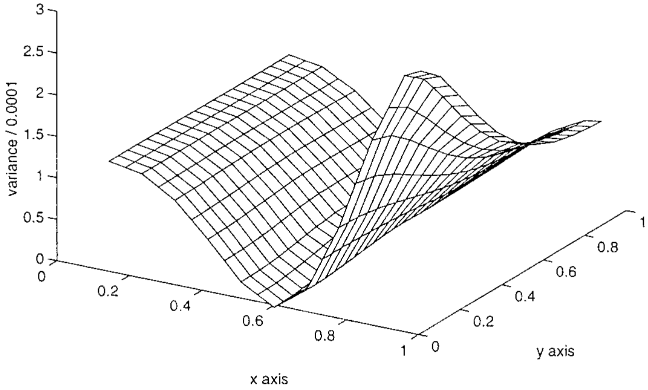

FIG. 8. Pressure variance for anisotropic correlation lengths at $t = 1.0$.

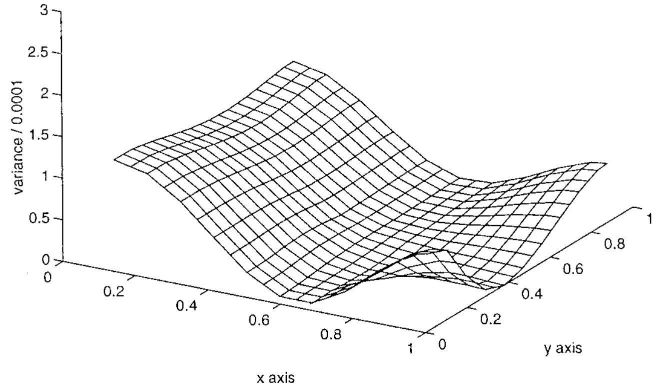

FIG. 10. Pressure variance for anisotropic correlation lengths at $t = 1.0$.

If we now take mean values on either side, then, since $\langle p_1 \rangle = 0$, we obtain a vector expression for the mean value of the flow given by

$$\langle \mathbf{Q} \rangle \simeq -\kappa_g \nabla p_0 - (\langle \kappa_1 \nabla p_1 \rangle + \langle \kappa_2 \rangle \nabla p_0 + \kappa_g \nabla \langle p_2 \rangle). \quad (66)$$

The covariance of the flow may be written

$$\begin{aligned} \mathrm{Cov}_q \simeq \langle \kappa_1 \kappa_1 \rangle (\nabla p_0)^2 + 2\kappa_g \nabla p_0 \cdot \langle \kappa_1 \nabla p_1 \rangle \\ + (\kappa_g)^2 \langle (\nabla p_1) \cdot (\nabla p_1) \rangle. \end{aligned} \quad (67)$$

Using the computational results obtained by the methods described in the previous sections, we can now compute the first two statistical moments for the flow. These only require statistical information for the pressure which is already available. Both these terms can then be used to calculate the mean of the net present value and its statistical moments up to second order.

It is fairly straightforward to approximate Eq. (64) with a central difference approximation so that the flow at the point $(i\,\Delta x, j\,\Delta y)$ can be written

$$\mathbf{Q_{ij}} = -k_{ij} \nabla_h p_{ij}. \quad (68)$$

The equation for the mean value of the flow then takes the form

$$\langle \mathbf{Q_{ij}} \rangle \simeq -\kappa_{ij}^g \nabla_h p_{ij}^0 - (\langle \kappa_{ij}^1 \nabla_h p_{ij}^1 \rangle + \langle \kappa_{ij}^2 \rangle \nabla_h p_{ij}^0 + \kappa_{ij}^g \nabla \langle p_{ij}^2 \rangle), \quad (69)$$

and the equivalent covariance term is

$$\begin{aligned} \mathrm{Cov}_{q_{ij}} \simeq \langle \kappa_{ij}^1 \kappa_{ij}^1 \rangle (\nabla_h p_{ij}^0)^2 + 2\kappa_{ij}^g \nabla_h p_{ij}^0 \cdot \langle \kappa_{ij}^1 \nabla_h p_{ij}^1 \rangle \\ + (\kappa_{ij}^g)^2 \langle (\nabla_h p_{ij}^1) \cdot (\nabla_h p_{ij}^1) \rangle. \end{aligned} \quad (70)$$

These discretised forms for the statistical moments of the flow are used to calculate numerical approximations to the NPV. In the case of a standard probability distribution function for the permeability, similar results can be derived.

6.2. *Net Present Value*

To assess the net present value of the systems we are considering, we must treat the NPV as a time-dependent variable; that is, we define

$$\mathrm{NPV}(t) = \int_0^t \|\mathbf{Q}\| e^{-\delta s}\, ds, \quad (71)$$

where $\mathbf{Q}$ is the flow at a specified position, and let $t \to \infty$. Here $\|\cdot\|$ denotes the L_2 vector norm and δ is the discount factor. The mean value of the NPV can then be shown to be

$$\langle \mathrm{NPV} \rangle \simeq \int_0^t \|\langle \mathbf{Q_{ij}} \rangle\| e^{-\delta s}\, ds, \quad (72)$$

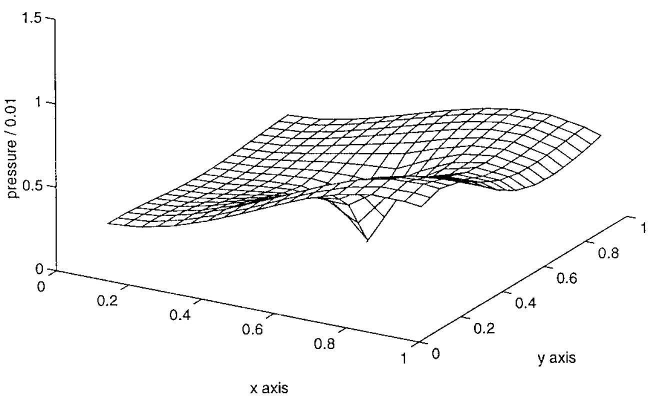

FIG. 9. Mean correction for anisotropic correlation lengths at $t = 1.0$.

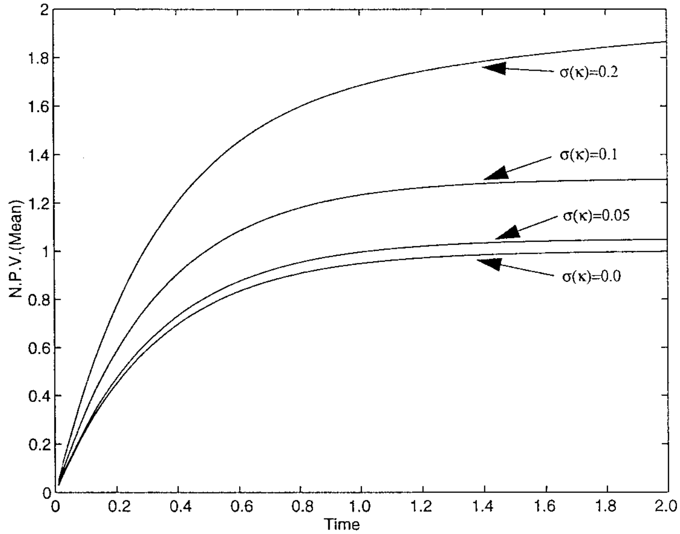

FIG. 11. Evolution of means of NPV for various σ_z^2.

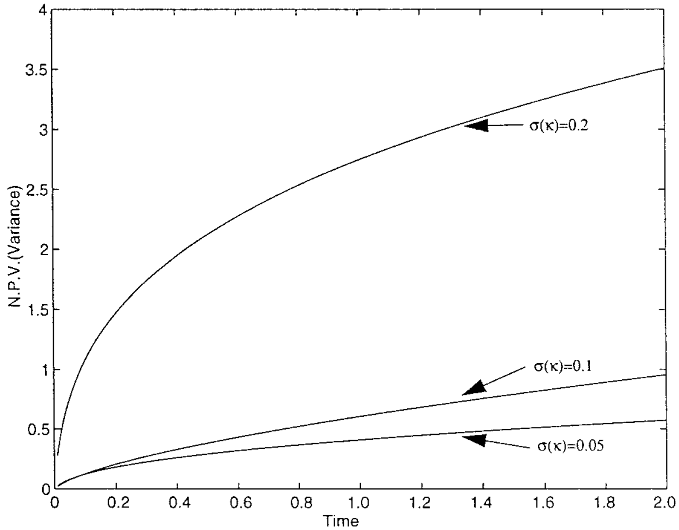

FIG. 12. Approximation of variance of NPVs for various σ_z^2.

to second-order accuracy, and an approximation to the second moment may be written as

$$\langle \mathrm{NPV}_2 \rangle \simeq \int_0^t \langle (Q_{ij} - \langle Q_{ij} \rangle)^2 \rangle e^{-\delta s}\, ds \equiv \int_0^t \mathrm{Cov}_{q_{ij}} e^{-\delta s}\, ds. \quad (73)$$

We are chiefly interested in how the mean value of the NPV compares with the deterministic solution, obtained by operating the numerical process on the mean value of the permeability field to give

$$\mathrm{N\tilde{P}V} = \int_0^t \|\tilde{\mathbf{Q}}\| e^{-\delta s}\, ds, \quad (74)$$

where

$$\tilde{\mathbf{Q}} = -\kappa_{ij}^g \nabla_h p_{ij}^0. \quad (75)$$

6.3. *Results*

We now give examples of risked values of a field that have been computed by the methods described here for finding the low order moments of the probability distribution function of the NPV. We take the same data as in Section 5.1 for the test problem. The discount factor is taken to be $\delta = 1.0$. Integrals are computed using the trapezoidal quadrature rule with time step $\Delta t = \frac{1}{100}$.

As before, we consider a single Fourier mode as the initial pressure condition in the reservoir, with no flow conditions around the boundary and zero forcing function. The region under investigation is a square of unit length, and all lengths and times are normalised. Using the single Fourier mode as the initial condition means that, in the case of a homogeneous geometric mean value for the permeability, the deterministic solution to Eq. (1) is given by Eq. (62).

We observe the values for the NPV over the time interval [0, 2] determined by the flow at the centre of the region. At this point the pressure p is deterministic and is held constant for all times t. These conditions correspond to those that hold at a well site. Figure 11 shows the various mean values for the NPV with different permeability variances, compared with the deterministic solution. The homogeneous geometric mean value of the permeability is $\kappa_g = 0.2$. In Fig. 12 the corresponding relative variances are shown for the NPV for the same permeability variances.

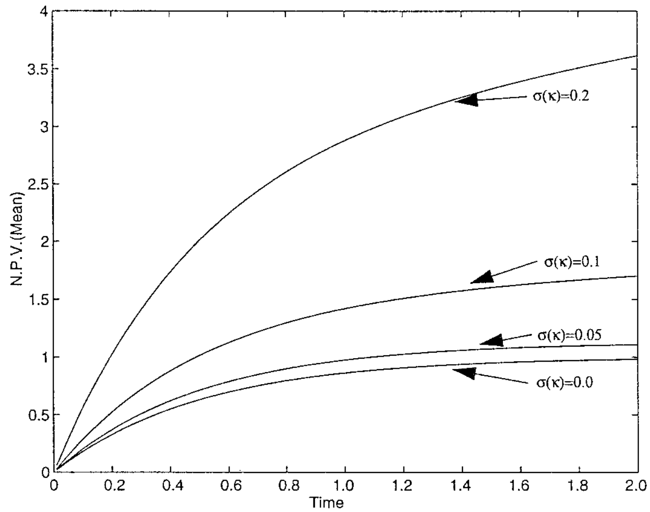

FIG. 13. Means of NPV for small κ_g for various σ_z^2.

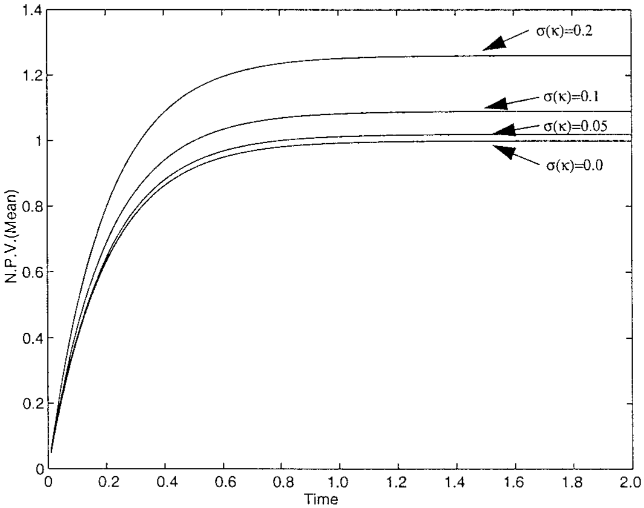

FIG. 14. Means of NPVs for large κ_g for various σ_z^2.

In Fig. 13, we show the equivalent plots in the case of a smaller permeability mean. Here, $\kappa_g = 0.1$. In Fig. 14, we show the plots of the mean of the NPV for a larger mean permeability field with $\kappa_g = 0.4$.

In Fig. 11 we can see that the mean values for the NPVs corresponding to the smaller values of the permeability field seem to converge to a similar order of magnitude, but to a significantly different value from the deterministic solution ($\sigma_z = 0.0$). The value for the case where the covariance of the permeability field is large with respect to its mean seems not to show convergence over the specified time period.

This effect is repeated in Figs. 13 and 14, with significant convergence being shown in Fig. 14, where the mean is always larger than the deterministic value of the NPV.

7. CONCLUSIONS

In this paper we establish a new method for computing the statistical moments of the probability distribution of a reservoir pressure field directly from statistical data describing the stochastic properties of the reservoir, such as permeability and porosity. We show also that the probability distributions of the flow and the net present value (NPV) of the field can be assessed from these results. The advantage of this method is that it requires only one solution of the field equations, in contrast with the more usual Monte-Carlo procedure where many such solutions are required.

The proposed method uses a perturbation expansion about the mean of the input parameters to derive a hierarchical system of numerical equations approximating the moments of the stochastic variables. The feasibility of this approach is demonstrated for a simple example of one phase flow in a two dimensional reservoir where the permeability field is characterised by its mean value and autocorrelation function and is assumed to be of lognormal form. Simple explicit finite difference schemes are used to approximate the pressure and flow equations. Second-order approximations to the mean and variance of the pressure field are calculated and the risked value of the field is estimated for various statistical descriptions of the permeability field. The results indicate that the estimated mean of the NPV varies significantly with the variance of the permeability field.

These results demonstrate that the direct approach described here can be used effectively to assess the potential of reservoirs with uncertain data. Further studies are needed to improve the efficiency and range of applicability of the process. The limitations imposed by the stability conditions can easily be removed by applying implicit difference schemes to obtain the numerical approximations. Efficiency could be improved by reducing the computation of the cross-correlation terms only to those making significant contributions to the moments.

The approach presented here is being extended to uncertain nonlinear multiphase flow problems. In these cases the method is expected to be particularly competitive, because the equations for the higher moments are linear and can be solved rapidly and efficiently, in contrast to Monte-Carlo methods, which require repeated solution of the full nonlinear models. The procedure can also be applied to other physical systems modelled by partial differential equations with uncertain data.

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